First-moment distinguishability bounds

Norman Margolus*
Massachusetts Institute of Technology
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The usual quantum uncertainty bounds are based on second moments of the quantum amplitude distribution. Here we present similar bounds based on first moments, in which average energy and momentum define the maximum average temporal and spatial rates of distinguishable state change.

INTRODUCTION

For an isolated quantum system with average energy E and average momentum of magnitude p,

$$E\,\tau \geq \frac{h}{2} \tag{1}$$

and

$$p\,\lambda \geq \frac{h}{2}\,. \tag{2}$$

Here τ is the average time it takes to evolve between distinct states, and λ is the average distance the system must travel to transition between distinct states (ignoring the system's internal dynamics). We take the zero of the energy scale to be the ground state energy of a large system that encompasses all subsystems of interest.

The bounds (1) and (2) are similar to well known quantum uncertainty bounds that depend on second moments of the energy and momentum amplitude distributions [1, 2] but these depend instead on first moments, which are fundamental conserved quantities. Moreover, unlike earlier bounds that govern *minimum* time or distance between distinct states (some also based on first moments [3, 4]), (1) and (2) constrain the *average* separation in a series of mutually orthogonal states. For periodic systems, the new bounds imply new uncertainty relations between number and phase [12], relevant in optics [5–7].

The energy bound (1) generalizes a bound proven in [3]. The momentum bound (2) is a relativistic consequence of (1), and has not appeared before. These bounds are the best possible with the given average quantities (*cf.* [8]). We also derive related bounds on ground state energy and on the minimum distance a system must travel to transition between distinct states.

DISTINGUISHABILITY IN TIME

In [3] we show that, for a periodic quantum evolution with period T that passes through N mutually orthogonal states at a constant rate,

$$\frac{2(E-E_0)}{h} \geq \frac{N-1}{T}\,, \tag{3}$$

where E_0 is the lowest energy eigenvalue used in constructing the system's state. The left hand side is a measure of the width of the energy eigenfrequency distribution: twice the average half-width. The right hand side is the minimum frequency bandwidth for N distinct frequencies separated from each other by at least $1/T$ [13].

In this section we show that (3) holds regardless of whether or not the time between distinct states is constant, that (1) follows from (3), and that (3) also places a bound on the minimum energy of an isolated system.

Periodic orthogonal dynamics

We begin by writing the state of a finite-sized isolated system at time t as

$$|t\rangle = \sum_n a_n e^{-2\pi i \nu_n t} |E_n\rangle \tag{4}$$

where $\nu_n = E_n/h$, and the average energy is

$$E = \sum_n |a_n|^2 E_n\,. \tag{5}$$

If the evolution passes through a series of mutually orthogonal states $|t_k\rangle$ at times t_k, we must have

$$\langle t_m | t_k \rangle = \sum_n |a_n|^2 e^{2\pi i \nu_n (t_k - t_m)} = \delta_{mk}\,. \tag{6}$$

Since the time evolution of an isolated quantum system is approximately periodic [10], we will simplify our analysis by considering only periodic evolutions. If a system returns *exactly* to its initial state after a period T, then its state must be a superposition of energy eigenstates with eigenfrequencies $\nu_n = E_n/h$ that are all integer multiples of $1/T$. The states $|t\rangle$ and $|t+T\rangle$ will, however, differ only by an overall phase if we add a constant to all the ν_n. Moreover, we can assume here that all the ν_n that appear in (4) are distinct, since if ν_j and $\nu_{j'}$ were equal, we could set $|a_{j'}|^2 = 0$ and move its weight to $|a_j|^2$ without changing either the average energy (5) or the orthogonality times in (6). Thus we can assume that the eigenfrequencies are of the form

$$\nu_n = \nu_0 + n/T\,, \tag{7}$$

with n a non-negative integer. Note that if T is large enough, a subset of the ν_n of this form can closely approximate any discrete frequency spectrum bounded from below. Since allowing extra eigenfrequencies never increases the minimum energy of states we can construct that satisfy the constraints (6), we can derive universal bounds by allowing *all* non-negative integers in (7).

Constant rate dynamics

We know from [3] that, for an isolated system,

$$\frac{2(E-E_0)}{h} \geq \frac{1}{2\tau_{\min}} , \tag{8}$$

where $\tau_{\min}$ is the minimum time interval between any two orthogonal states traversed during the system's evolution. As with (3) the right hand side is a minimum frequency bandwidth: in this case there must be at least two frequencies, separated by at least $1/2\tau_{\min}$.

The smallest E is allowed by (8) when $\tau_{\min}$ is largest. For a periodic evolution passing through N distinct states in time T, $\tau_{\min}$ is largest when all intervals equal T/N (since not all intervals can be above average).

For mutually orthogonal states equally separated in time [3], $t_m = mT/N$ and, from (6) and (7),

$$\langle t_{k+m}|t_k\rangle = e^{2\pi i\nu_0 t_m}\sum_{n=0}^{\infty}|a_n|^2 e^{2\pi inm/N} . \tag{9}$$

For an energetically optimal evolution, $a_n = 0$ for $n \geq N$, since if a_{n+N} is nonzero we can get exactly the same set of orthogonality times with a lower average energy by setting $a_{n+N} = 0$ and increasing the amplitude of a_n. Then, since $\langle t_{k+m}|t_k\rangle = \delta_{m0}$, the $|a_n|^2$ for $0 \leq n \leq N-1$ are just the discrete Fourier transform of a Kronecker delta impulse, and so they all equal $1/N$. Thus the average frequency is just the midpoint of a frequency range of width $(N-1)/T$, which gives the bound (3).

This bound is generally achievable for macroscopic systems [3]. In a limit in which the energy spectrum is nearly continuous, it is achieved by an equally weighted superposition using a set of equally spaced energy eigenvalues.

Comparison with a weaker bound

Since $\tau_{\min} \leq T/N$, substituting T/N for $\tau_{\min}$ in (8) directly yields the bound

$$\frac{2(E-E_0)}{h} \geq \frac{N}{2T} . \tag{10}$$

This bound agrees with (3) for $N = 2$, which is the only case for which (8) is achievable [8]. There is, however, a factor of 2 difference between (3) and (10) for large N, where (3) is achievable but (10) is not.

Non-constant rate dynamics

The argument that (8) implies that the minimum energy corresponds to maximum $\tau_{\min}$ and hence equal intervals, is only rigorous when (8) is achievable. We also point out, though, that making the intervals equal constrains the energy minimization problem the least.

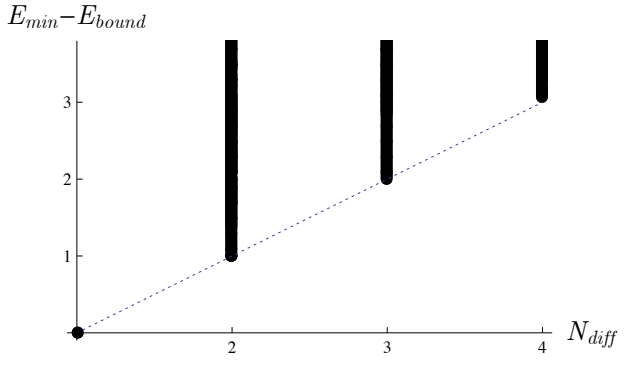

FIG. 1: Minimum energy $E_{\min}$ compatible with each of a large set of sample evolutions, where N_{diff} is the number of different interval lengths between consecutive distinct states in a given sample. Energy is in units of $h/2T$, and E_{bound} is the minimum energy allowed by the bound (3).

If all intervals are equal, then $N-1$ distinct constraint equations (6) must equal zero. Otherwise there are more constraints. Suppose, for example, that all but one of the intervals are equal. Then $2(N-1)$ constraint equations (6) must equal zero, except in the special case where the size of the exceptional interval is an integer multiple of the others. If the integer is 2, then only N equations must equal zero. In this case the constraints are exactly the same as for the case with $N+1$ equal intervals, and so the bound (3) for $N+1$ intervals applies.

This argument correctly predicts the minimum energy seen with two different interval lengths ($N_{\text{diff}} = 2$) in numerical experiments, depicted in Figure 1. To construct this diagram, a million sets of t_k's were generated stochastically, with each set involving only 2, 3, or 4 different interval lengths between adjacent t_k's (but with up to N=20 t_k's). For each set of t_k's, a finite set of a_n's that satisfy (6) while minimizing energy was determined using linear programming [14], and the resulting energy is plotted. Only integer interval lengths were used, to make it easy to ensure that the cases with the fewest constraints—which depend on integer ratios of interval lengths—were well sampled.

Generalizing the argument given for $N_{\text{diff}} = 2$ predicts that the minimum energy should grow quadratically with N_{diff} [15]. This prediction agrees with experiment when N is large compared to N_{diff}, but when their sizes are similar almost-equal intervals give the lowest energy, and in fact a linear dependence is observed [16]. For the values of N_{diff} shown in Figure 1 the deviation from the quadratic bound comes entirely from cases with $N < 8$.

Systems and subsystems

The bound (3) constrains how much greater the energy of a temporarily isolated subsystem must be than the ground state energy of a larger system it is part of.

To see this, first note that the bound (3) becomes (1) if $E_0 = h/2T$. We choose the zero of our energy scale so that this is true for the ground state energy E_{0L} and period T_L of the larger system. For $T_L \to \infty$ this is equivalent to setting $E_{0L} = 0$.

Now suppose that while a subsystem is temporarily isolated from the rest of the system it follows an evolution that, if it remained isolated long enough, would be periodic with period T_s. If the evolution has average energy E_s, lowest energy eigenvalue E_{0s}, and average time between distinct states $\tau_s = T_s/N_s$, then from (3),

$$E_s \geq \frac{h}{2\tau_s} + \left(E_{0s} - \frac{h}{2T_s} \right). \tag{11}$$

If the larger system achieves the bound (1), then the portion of the total energy temporarily located in the isolated subsystem must contribute its proportionate share to the total *constant* rate of state change, and no more. To guarantee that (1) can never be violated the term in parentheses in (11) must not be negative, and so

$$E_{0s} T_s \geq \frac{h}{2} \,. \tag{12}$$

This bound is achieved, for example, by the ground state energy of a simple harmonic oscillator.

Shorter averages

The average separation τ that appears in (1) can be computed over an interval Δt that is shorter than the system's full period, as long as $\Delta t \gg \tau$.

To see this we bound the energy E of the system's entire evolution by that needed for the interval Δt, by imagining the interval's separation times constitute a periodic evolution. The frequency bandwidth needed to describe the entire evolution won't be less than that needed for this portion. From (3), $2(E - E_0)/h \geq (N_\Delta - 1)/N_\Delta \tau_\Delta$, where N_Δ and τ_Δ refer to the interval Δt. This yields (1) to desired accuracy if $N_\Delta = \Delta t/\tau_\Delta$ is large enough.

DISTINGUISHABILITY IN SPACE

An isolated system in an energy eigenstate will never transition to an orthogonal state when viewed in its rest frame. Viewed in any other frame, however, the system will pass through a sequence of mutually orthogonal states due only to its net motion. For any isolated system, we define λ to be the average distance between states that are distinct due to net motion, and construct a bound that factors out internal motion.

Relativistic analysis

To simplify formulas in this analysis, we'll choose units in which $h = 2$. Consider an isolated system with net momentum of magnitude p that achieves the maximum (constant) average rate of orthogonal change for its energy. Now apply the energy bound (1) in two different relativistic frames. In the laboratory frame, the number of orthogonal states traversed by the system in moving freely between two events separated by distance Δx and time Δt is $\Delta t/\tau = E\Delta t$. In the rest frame the number of orthogonal states traversed is $E_r \Delta t_r$. Since none of the state changes in the rest frame are due to net motion of the system, the difference

$$E\Delta t - E_r \Delta t_r = p\Delta x \tag{13}$$

counts only the changes that are due to net motion (and in fact defines what we mean by this). Since the number of orthogonal states traversed due to a net motion of a distance Δx can also be written as $\Delta x/\lambda$, (2) is achieved.

Fictitious dynamics analysis

Another way to isolate the contribution to orthogonal change due to net motion is by imagining that we suddenly replace the dynamics of a freely moving system with a fictitious dynamics that shifts the system state with constant velocity $\vec{v}$ in the direction of the total average momentum $\vec{p}$, without any other time evolution. We can then use energy bounds on orthogonality in time to bound orthogonality in space due only to this motion.

The hamiltonian that generates a shift at velocity $\vec{v}$ is

$$\mathsf{H}_\star = \vec{v} \cdot \vec{\mathsf{p}} \,. \tag{14}$$

For this fictitious dynamics, the average energy $E_\star = vp$, where p is the magnitude of the average momentum of the original system (since $\vec{\mathsf{p}}$ is the momentum operator and, except for a shift, the state is unchanged). This *shift energy* depends only on the net momentum, and so it leaves out all energy associated with momenta that cancel (*i.e.,* the internal dynamics that has been suspended). The orthogonal changes due only to the shift dynamics are separated in space by $\lambda = v\tau$, and so $E_\star \tau = p\lambda$ and (1) implies (2). A similar argument bounds changes related to a system's overall rotation [17].

If we divide (13) by Δt we see that the shift energy $E_\star = vp$ defined here is just the relativistic energy excluding the internal energy of the freely moving system.

Component bounds

The shift dynamics (14) can be looked at as the sum of three independent orthogonal shifts. Thus each orthogonal component of the motion can be considered an isolated system that is separately bounded by (2). These component bounds add up as expected: the maximum number of orthogonal states traversed due to net motion of a distance l is $2pl/h$, and for $\vec{l}$ in the direction of the net momentum, $pl = \vec{p} \cdot \vec{l}$ and so the maximum number of distinct states is the sum of the component maxima.

Minimum orthogonal shift

Using the shift energy $E_\star = vp$ defined above to relate energy bounds to momentum bounds, (3), (8) and (12) have momentum counterparts. For example, define $\lambda_{\min} = v\tau_{\min}$. Since (12) requires $E_{0*} \geq 0$, (8) implies

$$p\lambda_{\min} \geq \frac{h}{4} \, . \tag{15}$$

This is similar to the bound in [4], which is based on the average of the magnitude of $\vec{\mathsf{p}}$, rather than the more physically relevant magnitude of the average. Here, as with λ in (2), we have defined $\lambda_{\min}$ to ignore state changes that aren't due to the system's net motion. This differs from [4], which gives an absolute bound on how far a given state must be shifted to reach a distinct state.

DISCUSSION

The maximum average rate at which an isolated system can traverse distinct states is given by its average energy: $1/\tau \leq 2E/h$. This bound is a consequence of Fourier analysis (the average bandwidth is bounded by the minimum possible bandwidth) plus a choice of zero for the energy scale (we take the ground state energy of a one-dimensional harmonic oscillator with period T to be $h/2T$). A corollary of the energy bound is that the magnitude of average momentum bounds traversal of states that are distinct due to net motion: $1/\lambda \leq 2p/h$.

Other tight bounds also follow from the premise that the average energy or momentum eigenfrequency bandwidth $2(\bar{\nu} - \nu_0)$ must be at least as large as the minimum bandwidth required by constraints on the separations in time or space between mutually orthogonal states. As a special case these bounds apply in communication theory, since expressing a scalar wavefunction as a superposition of momentum eigenstates is just resolving a complex-valued function into Fourier components.

All of these bounds apply equally well to model systems in which classical finite-state dynamics are recast as special cases of hamiltonian quantum dynamics [11]. In this restricted context, the finiteness of energy and momentum are consequences of the bounded Fourier spectra of continuous functions used to describe discrete space and time systems, and average energies and momenta count finite rates of change in the finite-state dynamics.

* Electronic address: nhm@mit.edu

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[12] For a periodic evolution with period T, let $\mathbb{N} = T\mathsf{H}/h$. If the system passes through N distinct states, (1) implies $\langle \mathbb{N} \rangle \geq N/2$. This becomes $\langle \mathbb{N} \rangle \overline{\Delta\theta} \geq \pi$ if $\Delta\theta = 2\pi\Delta t/T$ for states separated by Δt, making average $\overline{\Delta\theta} = 2\pi/N$.

[13] That the minimum bandwidth is $(N-1)/T$ is in accord with the bandwidth-time theorem [9].

[14] We chose the number of a_n's to be large enough to allow (6) to be satisfied, and verified that doubling the number of a_n's didn't change the minimum energy obtained.

[15] If $N_{\text{diff}} \ll N$ of the intervals are 1, 2, 3, etc. times larger than the rest, the set of differences between the t_k's is the same as for $N + N_{\text{diff}}(N_{\text{diff}} - 1)/2$ equally spaced states.

[16] To a precision of 15 decimal places, the minimum values of $E_{\min} - E_{\text{bound}}$ seen for $N_{\text{diff}} = \{1, 2, 3, 4\}$ are $\{0, 1, 2, 3.069609371327202\}$.

[17] We obtain the bound $J\theta \geq h/2$ if we substitute $\mathsf{H}_\star = \vec{\omega} \cdot \vec{\mathsf{J}}$ for (14) in the argument above. Here, though, θ is not simply the average angle between states that are distinct due to rotation; rotation also changes the dynamics.